

INVARIANTS OF OPEN BOOKS OF LINKS OF SURFACE SINGULARITIES

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ABSTRACT. If M is the link of a complex normal surface singularity, then it carries a canonical contact structure ξ_{can} , which can be identified from the topology of the 3-manifold M . We assume that M is a rational homology sphere. We compute the *support genus*, the *binding number* and the *norm* associated with the open books which support ξ_{can} , provided that we restrict ourself to the case of (analytic) Milnor open books. In order to do this, we determine monotonicity properties of the genus and the Milnor number of all Milnor fibrations in terms of the Lipman cone.

We generalize results of [3] valid for links of rational surface singularities, and we answer some questions of Etnyre and Ozbagci [7, section 8] regarding the above invariants.

1. INTRODUCTION

Let M be an oriented 3-dimensional manifold. By a result of Giroux [8] there is a one-to-one correspondence between open book decompositions of M (up to stabilization) and contact structures on M (up to isotopy). In [7] Etnyre and Ozbagci consider three invariants associated with a fixed contact structure ξ defined in terms of all open book decompositions supporting it:

- the *support genus* $sg(\xi)$ is the minimal possible genus for a page of an open book that supports ξ ;
- the *binding number* $bn(\xi)$ is the minimal number of binding components for an open book supporting ξ and that has pages of genus $sg(\xi)$;
- the *norm* $n(\xi)$ of ξ is the negative of the maximal (topological) Euler characteristic of a page of an open book that supports ξ .

In the present article we determine and characterize completely the above invariants under the following restrictions: M will be a *rational homology sphere which can be realized as the link of a complex surface singularity* $(S, 0)$. Moreover, we will restrict ourselves to the collection of those open book decompositions which can be realized as Milnor fibrations determined by some analytic germ (the so-called Milnor open books). Notice that by [5], all the Milnor open book decompositions define the same contact structure on M , the *canonical contact structure* ξ_{can} . This structure is also induced by *any* complex structure $(S, 0)$ realized on the topological type, and it can be characterized completely from the topology of M .

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Hence our results will be applied exactly for the canonical contact structure ξ_{can} , and for (analytic) Milnor open books, cf. section 5. The corresponding invariants are denoted by $sg_{an}(\xi_{can})$, $bn_{an}(\xi_{can})$ and $\mathbf{n}_{an}(\xi_{can})$.

The present article generalize results of [3] valid for links of rational surface singularities, and we answer some questions of [7, section 8] regarding the above invariants.

2. PRELIMINARIES

2.1. Invariants associated with a resolution. In what follows we assume that $(S, 0)$ is a complex normal surface singularity whose link is a rational homology sphere. Let $\pi : X \rightarrow S$ be a good resolution. We will denote by E_1, \dots, E_n the smooth irreducible components of the exceptional curve $E := \pi^{-1}(0)$ and by Γ its dual graph. By our assumption, each E_i has genus 0 and Γ is a tree.

Consider the free group $\mathcal{G} := H_2(X, \mathbb{Z})$ generated by the irreducible components of E , i.e. $\mathcal{G} = \{D = \sum_{i=1}^n m_i E_i \mid m_i \in \mathbb{Z}\}$. On \mathcal{G} there is a natural intersection pairing (\cdot, \cdot) and a natural partial ordering: $\sum_i m'_i E_i \leq \sum_i m''_i E_i$ if and only if $m'_i \leq m''_i$ for all i .

We denote the Lipman cone (semi-group) by

$$\mathcal{E}^+ = \{D \in \mathcal{G} \mid (D, E_i) \leq 0 \text{ for any } i\}.$$

It is known (see e.g. [2, 10]) that if $D = \sum m_i E_i \in \mathcal{E}^+$ then $m_i \geq 0$ for all i , and $m_i > 0$ for all i whenever $D \in \mathcal{E}^+ \setminus \{0\}$. Moreover, $\mathcal{E}^+ \setminus \{0\}$ admits a unique minimal element (the so-called Artin, or fundamental cycle), denoted by Z_{min} .

The definition of \mathcal{E}^+ is motivated by the following fact. Let $f : (S, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of an analytic function. Then the divisor $(\pi^*(f))$ in X of $f \circ \pi$ can be written as $D_\pi(f) + S_\pi(f)$, where $D_\pi(f)$, called the compact part of $(\pi^*(f))$, is supported on E , and $S_\pi(f)$ is the strict transform by π of $\{f = 0\}$. The collection of compact parts (when f runs over $\mathcal{O}_{S,0}$) forms a semi-group too, it will be denoted by \mathcal{A}^+ . It is a sub-semi-group of \mathcal{E}^+ (since $(\pi^*(f)) \cdot E_i = 0$ and $(S_\pi(f) \cdot E_i) \geq 0$ for all i). The subset $\mathcal{A}^+ \setminus \{0\}$ also has a unique minimal element Z_{max} , the *maximal ideal divisor*. It is the divisor of the generic hyperplane section. By definitions $Z_{min} \leq Z_{max}$.

For rational singularities one has $\mathcal{A}^+ = \mathcal{E}^+$ (hence $Z_{max} = Z_{min}$ too). But, in general, these equalities do not hold. The fundamental cycle Z_{min} can be obtained by Laufer's (combinatorial) algorithm (cf. [9]), but the structure of \mathcal{A}^+ (and even of Z_{max} too) can be very difficult, it depends essentially on the analytic structure of $(S, 0)$.

2.2. (Milnor) open books. Assume that $f : (S, 0) \rightarrow (\mathbb{C}, 0)$ defines an isolated singularity. Let M be the link of $(S, 0)$ and $L_f := f^{-1}(0) \cap M \subset M$ the (transversal) intersection of $f^{-1}(0)$ with M . Then the Milnor fibration of f defines an open book decomposition of M with binding L_f . One has the following facts:

- (1) For any f , consider an embedded good resolution π of the pair $(S, f^{-1}(0))$. Then the strict transform $S_\pi(f)$ intersects E transversally, and the number of intersection points $(S_\pi(f), E_i)$ (i.e. the number of binding components associated with E_i) is exactly $-(D_\pi(f), E_i)$. Since the intersection form is negative definite, the collection of binding components $\{(S_\pi(f), E_i)\}_{i=1}^n$ and $D_\pi(f) \in \mathcal{A}^+$ determine each other perfectly.

Moreover, by classical results of Stallings and Waldhausen, the (topological type of the) binding $L_f \subset M$ determines completely the open book up to an isotopy, provided that M is a rational homology sphere. ([6, page 34] provides two different arguments for this fact, one of them based on [4], the other one on [14]. For counterexamples for the statement in the general situation, see e.g. [11].)

Notice that the classification of all the (Milnor) open books associated with a *fixed analytic type* of $(S, 0)$ and analytic germs $f \in \mathcal{O}_{S,0}$ can be a very difficult problem (in fact, as difficult as the determination of \mathcal{A}^+).

- (2) Therefore, from a topological points of view, it is more natural to consider the open books of all the analytic germs associated with *all the analytic structures* supported by the topological type of $(S, 0)$.

Notice that for a fixed topological type of $(S, 0)$, in any (negative definite) plumbing graph of M one can also define the cone \mathcal{E}^+ . The point is that for any non-zero element D of \mathcal{E}^+ there is a *convenient* analytic structure on $(S, 0)$ and an analytic germ f , such that the plumbing graph can be identified with a dual resolution graph (which serves as an embedded resolution graph for the pair $(S, f^{-1}(0))$ too), and D is the compact part $D_\pi(f)$, see [13, 12]. Hence, changing the analytic structure of $(S, 0)$, we fill by the collections \mathcal{A}^+ all the semi-group \mathcal{E}^+ .

In particular, for any $Z \in \mathcal{E}^+ \setminus \{0\}$, there is an open book decomposition (well-defined up to an isotopy) realized as Milnor open book (by a convenient choice of the analytic objects).

- (3) For any fixed analytic type $(S, 0)$, the open book associated with Z_{max} is the Milnor fibration of the generic hyperplane section, in particular this open book is (resolution) graph-independent. Similarly, for a fixed topological type of $(S, 0)$, the open book associated with Z_{min} is also graph-independent. It depends only on the topology of the link.

2.3. Invariants of Milnor open books. Let us fix M , a plumbing (or, a dual resolution) graph Γ . Let us consider a Milnor open book associated with an element $Z \in \mathcal{E}^+ \setminus \{0\}$, cf. (2.2). In the sequel we will consider the following numerical invariants of it:

- (1) The number of binding components $\beta(Z)$ is given by $-(Z, E)$ (which is ≥ 1).
- (2) Let F be the fiber of the open book. It is an oriented connected surface with $-(Z, E)$ boundary components. Let $g(Z)$ be its genus (the so-called page-genus of the open book) and $\mu(Z)$ be the first Betti-number of F (the so-called Milnor number). Clearly:

$$(2.3.1) \quad \mu(Z) = 2 \cdot g(Z) - 1 + \beta(Z) = 2 \cdot g(Z) - 1 - (Z, E) \geq 2g(Z).$$

We will also write $\nu_i = (E_i, E - E_i)$, the number of components of $E - E_i$ meeting E_i .

2.4. The ‘monotoneity’ property. The main results of the next sections targets the ‘monotoneity’ property of invariants listed in (2.3).

Definition 2.4.1. Assume that for any resolution π of $(S, 0)$ one has a map $I_\pi : \mathcal{E}^+ \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$. We say that $I = \{I_\pi\}_\pi$ is monotone if for any two cycles $Z_i \in \mathcal{E}^+ \setminus \{0\}$ ($i = 1, 2$) with $Z_1 \leq Z_2$ one has $I_\pi(Z_1) \leq I_\pi(Z_2)$ for any π .

Remark 2.4.2. Assume that the collection of invariants $\{I_\pi\}_\pi$ can be transformed into (or comes from) an invariant I which associates with any (Milnor) open book \mathbf{m} of the link a non-negative integer. For any fixed analytic type, let \mathbf{m}_{\max} be the Milnor open book associated with Z_{\max} (considered in any resolution). Similarly, for any topological type, let \mathbf{m}_{\min} be the Milnor open book associated with Z_{\min} (in any resolution of an analytic structure conveniently chosen); cf. (2.2)(3).

Then, whenever $\{I_\pi\}_\pi$ is monotone, one has automatically the next consequences:

- (1) Fix an analytic singularity $(S, 0)$ and consider all the Milnor open books associated with all isolated holomorphic germs $f \in \mathcal{O}_{S,0}$. Then the minimum of integers $I(\mathbf{m})$ of all these Milnor open books \mathbf{m} is realized by the generic hyperplane section, i.e. by $I(\mathbf{m}_{\max})$.
- (2) Fix a topological type of a normal surface singularity, and consider the open books associated with all the isolated holomorphic germs of all the possible analytic structures supported by the fixed topological type. Then the minimum of all integers $I(\mathbf{m})$ of all these Milnor open books \mathbf{m} is realized by the open book associated with the Artin cycle, i.e. by $I(\mathbf{m}_{\min})$.

3. THE MONOTONEITY OF THE GENUS

3.1. The relation between the genus and the Euler-characteristic. For any fixed graph Γ , we consider the ‘canonical cycle’ $K \in \mathcal{G} \otimes \mathbb{Q}$ defined by the (adjunction formulas) $(K + E_i, E_i) + 2 = 0$ for all i . Then the (holomorphic) Euler-characteristic of any element $D \in \mathcal{G}$ is given by

$$(3.1.1) \quad \chi(D) := -\frac{1}{2}(D, D + K) \in \mathbb{Z}.$$

Proposition 3.1.2. *Fix $Z \in \mathcal{E}^+ \setminus \{0\}$. Then*

$$(3.1.3) \quad g(Z) = 1 + (Z, E) + \chi(-Z).$$

Proof. For any $1 \leq i \leq n$ consider $k_i := -(Z, E_i)$ (the number of binding components associated with E_i). Write also $Z = \sum_i m_i E_i$. Then by the A’Campo’s formula (cf. [1]) $1 - \mu = \sum_i (2 - \nu_i - k_i) m_i$. Then use (2.3.1) and (3.1.1). \square

Remark 3.1.4. Since $\chi(-Z) + \chi(Z) + Z^2 = 0$, one also has $g(Z) = 1 + (Z, E - Z) - \chi(Z)$. Since for any $Z \in \mathcal{E}^+ \setminus \{0\}$ one gets $Z \geq E$, one has $(Z, E - Z) \geq 0$ too. In particular:

$$(3.1.5) \quad g(Z) \geq 1 - \chi(Z).$$

Recall that *rational* singularities are characterized by $\chi(Z_{\min}) = 1$ [2]. If additionally, $(S, 0)$ is a *minimal* (i.e. if $Z_{\min} = E$), then $g(Z_{\min}) = 0$. For arbitrary rational germs one has $g(Z_{\min}) = (Z_{\min}, E - Z_{\min}) \geq 0$. This number, in general, might be non-zero: e.g. in the case of the E_8 -singularity it is 1. Considering arbitrary singularities, $\chi(Z_{\min})$ tends to $-\infty$ as the complexity of the topological type of the germ increases, hence by (3.1.5) $g(Z_{\min})$ tends to infinity too.

3.2. The “virtual genus” and its positivity. The formula (3.1.3) motivates the following definition. For $D = \sum_i m_i E_i \in \mathcal{G}$, let $|D|$ be the support $\sum_{i:m_i \neq 0} E_i$ of D and $\#(D)$ the number of connected components of $|D|$.

Definition 3.2.1. For $D \in \mathcal{G}$, $D \geq 0$, we define the “virtual genus” of D by

$$(3.2.2) \quad g(D) = \#(D) + (D, |D|) + \chi(-D).$$

Since for any $Z \in \mathcal{E}^+ \setminus \{0\}$ one has $|Z| = E$, and E is connected, (3.2.2) extends (3.1.3). Moreover, for any such $Z \in \mathcal{E}^+ \setminus \{0\}$, by its definition, $g(Z) \geq 0$.

Theorem 3.2.3. The virtual genus of any $D \in \mathcal{G}$, $D \geq 0$, is positive: $g(D) \geq 0$.

Proof. Assume that the statement is not true at least for one such a cycle. Since $g(E_i) = 1 + E_i^2 + \chi(-E_i) = 0$, there exist a minimal cycle $D > 0$ with $g(D) < 0$. Clearly, we can assume that $|D|$ is connected (and replacing Γ by its subgraph supported on $|D|$) that $|D| = E$. Write $D = \sum_i m_i E_i$. Hence we have:

$$(3.2.4) \quad 1 + (D, E) + \chi(-D) < 0.$$

and, using the notation $\#_i$ for the number of components of $|D - E_i|$:

$$(3.2.5) \quad \#_i + (D - E_i, |D - E_i|) + \chi(-D + E_i) \geq 0$$

for all E_i . Since $\chi(A + B) = \chi(A) + \chi(B) - (A, B)$, the two inequalities can easily be compared. Indeed, first assume that $m_i = 1$ for some i . Then $|D - E_i| = E - E_i$ and $\#_i = \nu_i$, hence (3.2.4) and (3.2.5) contradict each other. Therefore, $m_i \geq 2$ for all i . In that case, $|D - E_i| = E$ and $\#_i = 1$, hence (3.2.4) and (3.2.5) lead to $(D - E, E_i) \geq 0$ for all i . Hence $(D - E, D - E)$ is also non-negative by summation. Since the intersection form is negative definite, this implies $D = E$. This contradicts the fact that D is non-reduced (and also with the fact that $g(E) = 0$). \square

3.3. The monotoneity of the genus. The main result of this section is the following inequality:

Theorem 3.3.1. Consider two cycles Z and $Z + D$, where $Z \in \mathcal{E}^+ \setminus \{0\}$ and $D \in \mathcal{G}$, $D \geq 0$. Then the (virtual) genera satisfy $g(Z) \leq g(Z + D)$.

Proof. By (3.1.3), one has

$$\begin{aligned} g(Z + D) - g(Z) &= (D, E) + \chi(-D) - (D, Z) \\ &= g(D) + (D, E - |D|) - \#(D) - (D, Z). \end{aligned}$$

If $|D| = E$ then $\#(D) = 1$ and $-(D, Z) \geq 1$ (otherwise we would have $(Z, E_i) = 0$ for all i , or $Z = 0$). If $|D| < E$, then $-(D, Z) \geq 0$ and $(D, E - |D|) \geq (|D|, E - |D|) \geq \#(D)$ by the connectivity of Γ . Hence, in both cases, the right-hand side is $\geq g(D)$. Since $g(D) \geq 0$ by (3.2.3), the inequality follows. \square

Corollary 3.3.2. The genus is monotone: for any Z_1 and Z_2 from $\mathcal{E}^+ \setminus \{0\}$ with $Z_1 \leq Z_2$ one has $g(Z_1) \leq g(Z_2)$. In particular, the statements of (2.4.2) also hold.

4. THE MILNOR NUMBER AND THE NUMBER OF BOUNDARY COMPONENTS

4.1. The monotoneity of the Milnor number. If one combines (2.3.1) and (3.1.3), one gets for any $Z \in \mathcal{E}^+ \setminus \{0\}$:

$$(4.1.1) \quad \mu(Z) = 1 + (Z, E) + 2 \cdot \chi(-Z)$$

$$(4.1.2) \quad = g(Z) + \chi(-Z).$$

Again, we extend the above formula (in a compatible way with (4.1.2)) for any $D \geq 0$ by considering the ‘virtual Milnor number’ $\mu(D)$ as $g(D) + \chi(-D)$, defined via the virtual genus $g(D)$.

Clearly, $\mu(Z) \geq 0$ for any $Z \in \mathcal{E}^+ \setminus \{0\}$, since $\mu(Z)$ stays for a Betti number. Moreover, for any rational graph Γ , one has $\min \chi = 0$, hence for them the virtual invariants satisfy $\mu(D) \geq g(D) \geq 0$ too. The next theorem generalizes this for a general Γ .

Theorem 4.1.3. *Set $D \in \mathcal{G}$ with $D \geq 0$. Then the following inequalities hold:*

- (1) $\chi(-D) \geq 0$;
- (2) $\mu(D) \geq g(D) \geq 0$;
- (3) $\mu(Z + D) \geq \mu(Z)$ for any $Z \in \mathcal{E}^+ \setminus \{0\}$.

Proof. The proof of (1) is well-known for specialist, for the convenience of the reader we provide it. We claim that for any $D > 0$ there exists at least one E_i with $E_i \leq D$ such that $\chi(-D + E_i) \leq \chi(-D)$. This by induction shows that $\chi(-D) \geq 0$. The proof of the claim runs as follows. Assume that it is not true for some $D > 0$. Then for any E_i from its support one has $\chi(-D + E_i) \geq \chi(-D) + 1$. This is equivalent with $(D, E_i) \geq 0$, hence by summation one gets $D^2 \geq 0$. This implies $D = 0$, a contradiction.

(2) follows from (4.1.2), part (1) and (3.2.3). For (3) notice that by (4.1.2)

$$\mu(Z + D) - \mu(Z) = g(Z + D) - g(Z) + \chi(-D) - (Z, D).$$

Notice that $g(Z + D) \geq g(Z)$ by (3.3.1), $\chi(-D) \geq 0$ by (1), and $-(Z, D) \geq 0$ since $Z \in \mathcal{E}^+$. \square

Corollary 4.1.4. *The Milnor number is monotone: for any Z_1 and Z_2 from $\mathcal{E}^+ \setminus \{0\}$ with $Z_1 \leq Z_2$ one has $\mu(Z_1) \leq \mu(Z_2)$. In particular, the statements of (2.4.2) also hold for μ .*

4.2. The number of binding components. Recall that the number of binding components of the open book associated with some $Z \in \mathcal{E}^+ \setminus \{0\}$ is $\beta(Z) = -(Z, E)$. We wish to understand the variation of this number in the realm of (Milnor) open books with page-genus fixed. In order to do this, let us consider the following subsets of \mathcal{E}^+ :

$$\mathcal{E}_{min}^+ := \{Z \mid g(Z) = g(Z_{min})\}, \text{ and } \mathcal{E}_{g=a}^+ := \{Z \mid g(Z) = a\},$$

where $a \in \mathbb{Z}$. Since $\mu(Z) - \beta(Z) = 2g(Z) - 1$, we get:

Lemma 4.2.1. *For any a , the restrictions of μ and β to $\mathcal{E}_{g=a}^+$ take their minima on the same elements of $\mathcal{E}_{g=a}^+$. In particular, the restriction of μ (resp. of β) on \mathcal{E}_{min}^+ is $\mu(Z_{min})$ (resp. $\beta(Z_{min})$).*

5. APPLICATION TO THE CANONICAL CONTACT STRUCTURE OF THE LINK

Our application targets the invariants $\text{sg}_{an}(\xi_{can})$, $\text{bn}_{an}(\xi_{can})$ and $\mathbf{n}_{an}(\xi_{can})$; for notations, see Introduction. Indeed, the previous results read as follows:

$$\begin{aligned}\text{sg}_{an}(\xi_{can}) &= g(Z_{min}); \\ \text{bn}_{an}(\xi_{can}) &= \beta(Z_{min}); \\ \mathbf{n}_{an}(\xi_{can}) &= \mu(Z_{min}) - 1.\end{aligned}$$

In particular,

$$\mathbf{n}_{an}(\xi_{can}) - \text{bn}_{an}(\xi_{can}) = 2 \cdot \text{sg}_{an}(\xi_{can}) - 2.$$

These facts answer some of the questions of [7], section 8, at least in the realm of Milnor open books.

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